

THE MOTIVE OF THE CLASSIFYING STACK OF THE ORTHOGONAL GROUP

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ABSTRACT. We compute the motive of the classifying stack of an orthogonal group in the Grothendieck ring of stacks over a field of characteristic different from two. We also discuss an application of this result to Donaldson-Thomas theory with orientifolds.

1. INTRODUCTION

The Grothendieck ring of stacks over a field k has been introduced by a number of authors [1], [6], [8], [13]. Denote this ring by $\hat{K}_0(Var_k)$. An algebraic group G defined over k is called special if any G -torsor over a k -variety is locally trivial in the Zariski topology. General linear, special linear and symplectic groups are special. Special orthogonal groups are not special in dimensions greater than two. Serre proved that special groups are linear and connected [11]. Over algebraically closed fields, the special groups were classified by Grothendieck [7].

For a special group G , the motive $[G]$ is invertible in $\hat{K}_0(Var_k)$ and its inverse is equal to the motive of the classifying stack BG . This naturally raises the problem of computing the motive of BG when the group G is not special. For finite group schemes, a number of examples were computed in [5]. The case of groups of positive dimension is more difficult. In [3] it was shown that $[BPGL_n] = [PGL_n]^{-1}$ for $n = 2$ or 3 with mild restrictions on the field k .

The main result of this paper, Theorem 3.7, computes the motive of the classifying stack of an orthogonal group over a field whose characteristic is not two. The result is that the motive is equal to the inverse of the motive of the split special orthogonal group of the same dimension. To prove Theorem 3.7 we first compute the motive of the variety of non-degenerate quadratic forms of fixed dimension. This motive was already computed in [2], using results of [9]. Our computation is different, relying on generating function techniques. Using Theorem 3.7 we are able to compute the motive of classifying stack of the special orthogonal groups in odd dimensions. We also describe an application of the result to Donaldson-Thomas theory with orientifolds.

Notation. We will work over a base field k with $\text{char}(k) \neq 2$. If n is a non-negative integer we denote by $[n]_{\mathbb{L}}$ the n th Gaussian polynomial in the Lefschetz motive \mathbb{L} . Explicitly,

$$[n]_{\mathbb{L}} = 1 + \mathbb{L} + \cdots + \mathbb{L}^{n-1}.$$

The Gaussian polynomials $[n]_{\mathbb{L}}!$ and $\begin{bmatrix} n \\ r \end{bmatrix}_{\mathbb{L}}$ are defined in the usual way. The class of the Grassmannian $Gr(r, n)$ in the ring $\hat{K}_0(Var_k)$ is then $\begin{bmatrix} n \\ r \end{bmatrix}_{\mathbb{L}}$.

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2. PRELIMINARIES

2.1. The Grothendieck ring of stacks. Fix a ground field k . Let $K_0(\text{Var}_k)$ be the Grothendieck ring of varieties over k . Its underlying abelian group is generated by symbols $[X]$, with X a k -variety, modulo the relations $[X] = [Y]$ if X and Y are isomorphic and

$$[X] = [X \setminus Z] + [Z]$$

if $Z \subset X$ is a closed subvariety. Cartesian product of varieties gives $K_0(\text{Var}_k)$ the structure of a commutative ring with identity $1 = [\text{Spec } k]$. The Lefschetz motive is $\mathbb{L} = [\mathbb{A}_k^1]$.

The Grothendieck ring of stacks, $\hat{K}_0(\text{Var}_k)$, is the dimensional completion of $K_0(\text{Var}_k)$ defined as follows [1]. Let $F^m \subset K_0(\text{Var}_k)[\mathbb{L}^{-1}]$ be the additive subgroup generated by those $\mathbb{L}^{-d}[X]$ with $\dim X - d \leq -m$. This defines a descending filtration of $K_0(\text{Var}_k)[\mathbb{L}^{-1}]$ and $\hat{K}_0(\text{Var}_k)$ is the completion with respect to this filtration.

It is shown in [1] that any Artin stack that is essentially of finite type, all of whose geometric stabilizers are linear algebraic groups, has a motivic class in $\hat{K}_0(\text{Var}_k)$.

Lemma 2.1 ([1, Lemma 2.5]). *Let \mathfrak{X} be an Artin stack (satisfying the conditions above) and let G be a linear algebraic group. Then any G -torsor $P \rightarrow \mathfrak{X}$ has a motivic class $[P] \in \hat{K}_0(\text{Var}_k)$. Moreover, if G is special, then $[P] = [\mathfrak{X}][G]$ in $\hat{K}_0(\text{Var}_k)$.*

In particular, if G is special, applying Lemma 2.1 to the universal G -torsor $\text{Spec } k \rightarrow BG$ shows that $[BG] = [G]^{-1}$. This equality is called the universal G -torsor relation.

More generally, if X is a variety acted on by a linear algebraic group G , then the quotient stack X/G has a class in $\hat{K}_0(\text{Var}_k)$. For any closed embedding $G \hookrightarrow GL_N$ there is an isomorphism of stacks $X/G \simeq (X \times_G GL_N)/GL_N$. Since GL_N is special, Lemma 2.1 implies that

$$[X/G] = \frac{[X \times_G GL_N]}{[GL_N]} \quad (1)$$

in $\hat{K}_0(\text{Var}_k)$.

2.2. Orthogonal groups. Assume that the ground field k is not of characteristic two. Let V be a finite dimensional vector space over k and let $Q : V \rightarrow k$ be a quadratic form. The radical of Q is the subspace of V defined by

$$\text{rad}_Q = \{v \in V \mid Q(v+w) = Q(v) + Q(w) \quad \forall w \in V\}.$$

The rank of Q is $\dim V - \dim \text{rad}_Q$. The quadratic form Q is called nondegenerate if $\text{rad}_Q = \{0\}$.

Given a nondegenerate quadratic form Q , denote by $O(Q)$ its group of isometries. If the field k is algebraically closed, then there is a unique nondegenerate quadratic form on k^n up to equivalence. The corresponding orthogonal group is unique up to isomorphism. If k is not algebraically closed, then there will in general exist inequivalent nondegenerate quadratic forms on k^n , leading to different forms of orthogonal groups.

For each $n \geq 1$, there is a canonical nondegenerate split quadratic form on k^n . Explicitly,

$$Q_{2r} = x_1x_2 + \cdots + x_{2r-1}x_{2r}$$

and

$$Q_{2r+1} = x_0^2 + x_1x_2 + \cdots + x_{2r-1}x_{2r}.$$

Define $O_n = O(Q_n)$ and $SO_n = SO(Q_n)$.

3. THE MOTIVE OF $BO(Q)$

3.1. Filtration of the space of quadratic forms. Recall that $\text{char}(k) \neq 2$.

Denote by $Quad_n \simeq \mathbb{A}_k^{\binom{n+1}{2}}$ the affine space of quadratic forms on k^n . The group GL_n acts on $Quad_n$ by change of basis. For each $0 \leq r \leq n$, let $Quad_{n,\leq r} \subset Quad_n$ denote the closed subvariety of quadratic forms whose rank is at most r . This gives an increasing filtration of $Quad_n$ by closed subvarieties. Interpreted in $K_0(Var_k)$, this implies the identity

$$\mathbb{L}^{\binom{n+1}{2}} = \sum_{r=0}^n [Quad_{n,r}] \quad (2)$$

with $Quad_{n,r}$ the subvariety of quadratic forms of rank r . Denote by $Gr(m, n)$ the Grassmannian of m -planes in k^n .

Proposition 3.1. *For each $0 \leq r \leq n$, the map*

$$\pi : Quad_{n,r} \rightarrow Gr(n-r, n), \quad Q \mapsto \text{rad}_Q$$

is a Zariski locally trivial fibration with fibres isomorphic to $Quad_{r,r}$.

Proof. Identify $Gr(n-r, n)$ with the quotient of the variety of $(n-r) \times n$ matrices of rank $n-r$ by the left action of GL_{n-r} . Fix coordinates x_1, \dots, x_n on k^n . Consider the $(n-r)$ -plane $k^{n-r} \subset k^n$ with coordinates x_1, \dots, x_{n-r} . A Zariski open set $U \subset Gr(n-r, n)$ containing k^{n-r} is given by the $(n-r) \times n$ matrices of the form

$$\begin{pmatrix} \mathbf{1}_{n-r} & B \end{pmatrix}$$

with $\mathbf{1}_{n-r}$ the $(n-r) \times (n-r)$ identity matrix and B an arbitrary $(n-r) \times r$ matrix. The plane k^{n-r} corresponds to the matrix $B = 0$. Note that

$$\begin{pmatrix} \mathbf{1}_{n-r} & B \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{n-r} & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1}_{n-r} & B \\ 0 & \mathbf{1}_r \end{pmatrix}.$$

Let $g_B = \begin{pmatrix} \mathbf{1}_{n-r} & B \\ 0 & \mathbf{1}_r \end{pmatrix} \in GL_n$, viewed as an automorphism of k^n .

Suppose that $Q \in \pi^{-1}(U)$. Then there exists a unique matrix $B(Q)$ such that $\text{rad}_Q = g_{B(Q)}(k^{n-r}) \subset k^n$. The quadratic form $g_{B(Q)} \cdot Q$ is the pullback of a nondegenerate quadratic form φ_Q in the variables x_{n-r+1}, \dots, x_n . A trivialization of π over U is then given by

$$\pi^{-1}(U) \rightarrow U \times Quad_{r,r}, \quad Q \mapsto (\text{rad}_Q, \varphi_Q).$$

This argument can be repeated, replacing k^{n-r} with the $(n-r)$ -plane with coordinates labelled by a $(n-r)$ -element subset $I \subset \{1, \dots, n\}$. This gives a Zariski open cover of $Gr(n-r, n)$ over which π trivializes. \square

Corollary 3.2. *The identity*

$$\mathbb{L}^{\binom{n+1}{2}} = \sum_{r=0}^n \begin{bmatrix} n \\ n-r \end{bmatrix}_{\mathbb{L}} [Quad_{r,r}]_{\mathbb{L}}$$

holds in the ring $K_0(Var_k)$.

Proof. It follows from Proposition 3.1 that $[Quad_{n,r}] = [Gr(n-r, n)][Quad_{r,r}]$. Since $[Gr(n-r, n)] = \left[\begin{smallmatrix} n \\ n-r \end{smallmatrix} \right]_{\mathbb{L}}$, the desired identity is implied by equation (2). \square

3.2. Solving the recurrence. In this section we will solve the recurrence relation for $[Quad_{n,n}]$ given in Corollary 3.2. In fact, the motives $[Quad_{n,r}]$ were already computed in [2, Theorem 13.5], where it was shown that $[Quad_{n,r}]$ satisfies a certain three step recurrence relation with coefficients in $\mathbb{Z}[\mathbb{L}]$. This recurrence relation, with \mathbb{L} replaced by q , was previously solved in [9] to find the number of \mathbb{F}_q -rational points of $Quad_{n,r}$. Hence $[Quad_{n,r}]$ is given by the same formula, with q replaced with \mathbb{L} . We present here an alternative computation of $[Quad_{n,n}]$, and therefore also $[Quad_{n,r}]$ by Proposition 3.1, using generating functions.

We form the exponential generating function for the motives $[Quad_{n,n}]$,

$$G(x) = \sum_{n \geq 0} \frac{[Quad_{n,n}]x^n}{[n]_{\mathbb{L}}!}.$$

Consider also the auxiliary generating functions

$$P_{\text{even}}(x) = \sum_{k \geq 0} \frac{x^{2k}}{[2k]_{\mathbb{L}}!} \prod_{i=1}^k (\mathbb{L}^{2k+1} - \mathbb{L}^{2i})$$

and

$$P_{\text{odd}}(x) = \sum_{k \geq 0} \frac{x^{2k+1}}{[2k+1]_{\mathbb{L}}!} \prod_{i=0}^k (\mathbb{L}^{2k+1} - \mathbb{L}^{2i}).$$

We will show that

$$G(x) = P_{\text{even}}(x) + P_{\text{odd}}(x),$$

thereby solving the recurrence relation.

Proposition 3.3. *Denote by $\exp_{\mathbb{L}}(x)$ the \mathbb{L} -deformed exponential series:*

$$\exp_{\mathbb{L}}(x) = \sum_{n \geq 0} \frac{x^n}{[n]_{\mathbb{L}}!}.$$

The following equality holds:

$$G(x) = \frac{\prod_{i \geq 1} (1 + (1 - \mathbb{L})x\mathbb{L}^i)}{\exp_{\mathbb{L}}(x)}.$$

Proof. To ease notation, set $\mathcal{Q}_n = [Quad_{n,n}]$. Using Corollary 3.2 we find that

$$\begin{aligned} G(x) &= \sum_{n \geq 0} \frac{\mathcal{Q}_n}{[n]_{\mathbb{L}}!} x^n \\ &= \sum_{n \geq 0} \left(\mathbb{L}^{\binom{n+1}{2}} - \sum_{r=0}^{n-1} \left[\begin{smallmatrix} n \\ n-r \end{smallmatrix} \right]_{\mathbb{L}} \mathcal{Q}_r \right) \frac{x^n}{[n]_{\mathbb{L}}!} \\ &= \sum_{n \geq 0} \left(\mathbb{L}^{\binom{n+1}{2}} - \sum_{r=0}^{n-1} \frac{[n]_{\mathbb{L}}!}{[n-r]_{\mathbb{L}}! [r]_{\mathbb{L}}!} \mathcal{Q}_r \right) \frac{x^n}{[n]_{\mathbb{L}}!} \\ &= \sum_{n \geq 0} \left(\mathbb{L}^{\binom{n+1}{2}} \frac{x^n}{[n]_{\mathbb{L}}!} - \sum_{r=0}^{n-1} \frac{\mathcal{Q}_r x^r}{[r]_{\mathbb{L}}!} \frac{x^{n-r}}{[n-r]_{\mathbb{L}}!} \right) \\ &= \sum_{n \geq 0} \mathbb{L}^{\binom{n+1}{2}} \frac{x^n}{[n]_{\mathbb{L}}!} - \sum_{n \geq 0} \sum_{r=0}^n \frac{\mathcal{Q}_r x^{n-r}}{[r]_{\mathbb{L}}! [n-r]_{\mathbb{L}}!} + \sum_{n \geq 0} \frac{\mathcal{Q}_n x^n}{[n]_{\mathbb{L}}!} \\ &= \sum_{n \geq 0} \mathbb{L}^{\binom{n+1}{2}} \frac{x^n}{[n]_{\mathbb{L}}!} - \exp_{\mathbb{L}}(x) G(x) + G(x). \end{aligned}$$

Recall that

$$[n]_{\mathbb{L}}! = \frac{(1 - \mathbb{L})(1 - \mathbb{L}^2) \cdots (1 - \mathbb{L}^n)}{(1 - \mathbb{L})^n}$$

so that we have

$$\sum_{n \geq 0} \mathbb{L}^{\binom{n+1}{2}} \frac{x^n}{[n]_{\mathbb{L}}!} = \sum_{n \geq 0} \frac{\mathbb{L}^{\binom{n+1}{2}} (1 - \mathbb{L})^n x^n}{(1 - \mathbb{L})(1 - \mathbb{L}^2) \cdots (1 - \mathbb{L}^n)}.$$

We make use of the identity

$$\prod_{i \geq 1} (1 + (1 - \mathbb{L})x\mathbb{L}^i) = \sum_{n \geq 0} \frac{\mathbb{L}^{\binom{n+1}{2}} (1 - \mathbb{L})^n x^n}{(1 - \mathbb{L})(1 - \mathbb{L}^2) \cdots (1 - \mathbb{L}^n)},$$

see [12, Proposition 1.8.6]. Hence

$$G(x) = \frac{\prod_{i \geq 1} (1 + (1 - \mathbb{L})x\mathbb{L}^i)}{\exp_{\mathbb{L}}(x)},$$

which completes the proof. \square

It will be convenient to make the change of variables $g(x) = G(\frac{x}{1-\mathbb{L}})$.

Proposition 3.4. *We have*

$$\begin{aligned} g(x) &= (1 - x) \prod_{k \geq 1} (1 - x^2 \mathbb{L}^{2k}) \\ &= (1 - x) \sum_{k \geq 0} \frac{(-1)^k x^{2k} \mathbb{L}^{k(k+1)}}{(1 - \mathbb{L}^2)(1 - \mathbb{L}^4) \cdots (1 - \mathbb{L}^{2k})}. \end{aligned}$$

Proof. We compute

$$\begin{aligned} \exp_{\mathbb{L}}(x) &= \sum_{n \geq 0} \frac{x^n}{[n]_{\mathbb{L}}!} \\ &= \sum_{n \geq 0} \frac{x^n (1 - \mathbb{L})^n}{(1 - \mathbb{L})(1 - \mathbb{L}^2) \cdots (1 - \mathbb{L}^n)} \\ &= \frac{1}{\prod_{i \geq 0} (1 - (1 - \mathbb{L})x\mathbb{L}^i)}, \end{aligned}$$

where the last equality is via [12, page 74]. The first assertion of the proposition now follows. The second follows from the first by [12, Proposition 1.8.6]. \square

Similarly, make the change of variables $p_{\text{even}}(x) = P_{\text{even}}(\frac{x}{1-\mathbb{L}})$ and $p_{\text{odd}}(x) = P_{\text{odd}}(\frac{x}{1-\mathbb{L}})$.

Proposition 3.5. *We have*

$$p_{\text{even}}(x) = \sum_{k \geq 0} \frac{(-1)^k x^{2k} \mathbb{L}^{k(k+1)}}{(1 - \mathbb{L}^2)(1 - \mathbb{L}^4) \cdots (1 - \mathbb{L}^{2k})}$$

and

$$p_{\text{odd}}(x) = \sum_{k \geq 0} \frac{(-1)^{k+1} x^{2k} \mathbb{L}^{k(k+1)}}{(1 - \mathbb{L}^2)(1 - \mathbb{L}^4) \cdots (1 - \mathbb{L}^{2k})}.$$

Proof. The generating function P_{even} can be rewritten as

$$P_{\text{even}}(x) = \sum_{k \geq 0} \frac{(1 - \mathbb{L})^{2k} x^{2k}}{(1 - \mathbb{L})(1 - \mathbb{L}^2) \cdots (1 - \mathbb{L}^{2k})} \prod_{i=1}^k (\mathbb{L}^{2k+1} - \mathbb{L}^{2i}).$$

Then we have

$$\begin{aligned}
p_{\text{even}}(x) &= \sum_{k \geq 0} \frac{x^{2k}}{(1 - \mathbb{L})(1 - \mathbb{L}^2) \cdots (1 - \mathbb{L}^{2k})} \prod_{i=1}^k (\mathbb{L}^{2k+1} - \mathbb{L}^{2i}) \\
&= \sum_{k \geq 0} \frac{x^{2k} \mathbb{L}^{k(k+1)}}{(1 - \mathbb{L})(1 - \mathbb{L}^2) \cdots (1 - \mathbb{L}^{2k})} \prod_{i=1}^k (\mathbb{L}^{2(k-i)+1} - 1) \\
&= \sum_{k \geq 0} \frac{(-1)^k x^{2k} \mathbb{L}^{k(k+1)}}{(1 - \mathbb{L}^2)(1 - \mathbb{L}^4) \cdots (1 - \mathbb{L}^{2k})}.
\end{aligned}$$

The other calculation is similar. \square

Corollary 3.6. *The following identity holds in $\hat{K}_0(\text{Var}_k)$:*

$$G(x) = P_{\text{even}}(x) + P_{\text{odd}}(x).$$

Proof. As $(1 - \mathbb{L})$ is a unit in $\hat{K}_0(\text{Var}_k)$ it suffices to show that

$$g(x) = p_{\text{even}}(x) + p_{\text{odd}}(x).$$

This follows from Propositions 3.4 and 3.5. \square

3.3. The main theorem. We now state the main result.

Theorem 3.7. *Let k be a field whose characteristic is not 2 and let $n \geq 1$. For any nondegenerate quadratic form Q on k^n , the following equality holds in $\hat{K}_0(\text{Var}_k)$:*

$$[BO(Q)] = \begin{cases} \mathbb{L}^{-r} \prod_{i=0}^{r-1} (\mathbb{L}^{2r} - \mathbb{L}^{2i})^{-1}, & \text{if } n = 2r + 1 \\ \mathbb{L}^r \prod_{i=0}^{r-1} (\mathbb{L}^{2r} - \mathbb{L}^{2i})^{-1}, & \text{if } n = 2r. \end{cases}$$

Proof. The subvariety $\text{Quad}_{n,n} \subset \text{Quad}_n$ is stable under the action of GL_n on Quad_n . Pick $Q \in \text{Quad}_{n,n}$. This gives rise to an orbit morphism $GL_n \rightarrow \text{Quad}_{n,n}$. Since $\pi : GL_n \rightarrow GL_n/O(Q)$ is a uniform categorical quotient [10, Theorem 1.1], the orbit morphism factors through a unique morphism $\psi : GL_n/O(Q) \rightarrow \text{Quad}_{n,n}$. We claim that ψ is an isomorphism.

Let \bar{k} be an algebraic closure of k . Base change gives a morphism

$$\bar{\pi} : GL_{n,\bar{k}} \rightarrow GL_n/O(Q) \times_k \bar{k}$$

which is a categorical quotient for the action of $O(Q)_{\bar{k}}$ on $GL_{n,\bar{k}}$. Here $GL_{n,\bar{k}}$ denotes the general linear group over \bar{k} while $O(Q)_{\bar{k}}$ denotes orthogonal group of the quadratic form $Q \times_k \bar{k}$ on \bar{k}^n . The universal property of categorical quotients implies

$$GL_n/O(Q) \times_k \bar{k} \simeq GL_{n,\bar{k}}/O(Q)_{\bar{k}}.$$

Using this isomorphism and applying base change to ψ gives

$$\bar{\psi} : GL_{n,\bar{k}}/O(Q)_{\bar{k}} \rightarrow \text{Quad}_{n,n} \times_k \bar{k}.$$

Since $\text{Quad}_{n,n} \times_k \bar{k}$ is homogeneous under the action of $GL_{n,\bar{k}}$ with stabilizer $O(Q)_{\bar{k}}$, the map $\bar{\psi}$ is an isomorphism. By faithfully flat descent it follows that ψ itself is an isomorphism.

Identifying $BO(Q)$ with the quotient stack $\text{Spec } k/O(Q)$, equation (1) gives

$$[BO(Q)] = \left[\frac{GL_n/O(Q)}{GL_n} \right] = \frac{[GL_n/O(Q)]}{[GL_n]} = \frac{[\text{Quad}_{n,n}]}{[GL_n]}.$$

Using Corollary 3.6, we read off from P_{even} and P_{odd} the equality

$$[Quad_{n,n}] = \begin{cases} \prod_{i=0}^r (\mathbb{L}^{2r+1} - \mathbb{L}^{2i}), & \text{if } n = 2r + 1 \\ \prod_{i=1}^r (\mathbb{L}^{2r+1} - \mathbb{L}^{2i}), & \text{if } n = 2r. \end{cases}$$

If $n = 2r + 1$, we have

$$\begin{aligned} \frac{[Quad_{2r+1,2r+1}]}{[GL_{2r+1}]} &= \frac{\prod_{i=0}^r (\mathbb{L}^{2r+1} - \mathbb{L}^{2i})}{\prod_{i=0}^{2r} (\mathbb{L}^{2r+1} - \mathbb{L}^i)} \\ &= \prod_{i=0}^{r-1} (\mathbb{L}^{2r+1} - \mathbb{L}^{2i+1})^{-1} \\ &= \mathbb{L}^{-r} \prod_{i=0}^{r-1} (\mathbb{L}^{2r} - \mathbb{L}^{2i})^{-1} \end{aligned}$$

which is the desired result. The calculation for n even is analogous. \square

Corollary 3.8. *Suppose that $n \geq 3$ and let Q be a nondegenerate quadratic form on k^n . Then $[BO(Q)] = [SO_n]^{-1}$. Moreover, if n is odd, then $[BSO(Q)] = [SO_n]^{-1}$.*

Proof. Since $n \geq 3$, the split group SO_n is semisimple. According to [1, Lemma 2.1],

$$[SO_{2r+1}] = \mathbb{L}^r \prod_{i=0}^{r-1} (\mathbb{L}^{2r} - \mathbb{L}^{2i}), \quad [SO_{2r}] = \mathbb{L}^{-r} \prod_{i=0}^{r-1} (\mathbb{L}^{2r} - \mathbb{L}^{2i}).$$

Comparing these expressions with Theorem 3.7 gives the first statement. Continuing, if Q is a nondegenerate quadratic form in odd dimensions there is an isomorphism $O(Q) \simeq \mu_2 \times SO(Q)$. It is shown in [5, Proposition 3.2] that $[B\mu_2] = 1$. Hence

$$[BO(Q)] = [B\mu_2 \times BSO(Q)] = [B\mu_2][BSO(Q)] = [BSO(Q)].$$

The second statement now follows from the first. \square

Since $PGL_2 \simeq SO_3$ over any field, Corollary 3.8 recovers the first part of [3, Theorem A] as a special case.

It follows from Corollary 3.8 that the universal torsor relations are satisfied for split special orthogonal groups in odd dimensions. In particular, the universal $SO_{2n+1}(\mathbb{C})$ -torsor relation holds. In [4, Theorem 2.2] it is shown that for any non-special connected reductive complex algebraic group G there exists a G -torsor $P \rightarrow X$ over a variety such that $[P]$ is not equal to $[G][X]$. Therefore, the universal G -torsor relation does not imply the general G -torsor relation, answering a question posed in [1, Remark 3.3]. In the recent paper [3] the groups $PGL_2(\mathbb{C})$ and $PGL_3(\mathbb{C})$ were also shown to answer this question.

3.4. Motivic Donaldson-Thomas theory with orientifolds. Suppose that G is a linear algebraic group and let R be a finite dimensional representation of G . Lemma 2.1 and the fact that general linear groups are special implies $[R/G] = [R][BG]$. If G is a product of general linear, symplectic and orthogonal groups, then $[BG]$, and hence $[R/G]$, can be computed explicitly using Theorem 3.7.

Exactly the above situation arises in the study of moduli stacks of self-dual representations of a quiver with involution [14]. In this case, G and R are naturally defined over finite fields of odd characteristic. As the motivic class $[R/G]$ is rational in \mathbb{L} , it follows that $[R/G]$ and the number of \mathbb{F}_q -rational points $\#[R/G](\mathbb{F}_q)$ agree under the substitution $\mathbb{L} \leftrightarrow q$. This implies that the orientifold Donaldson-Thomas

series with trivial stability, defined in [14], are in fact motivic. In view of [14, Theorem 3.2], this suggests that an analogous statement holds for arbitrary σ -compatible stability. The second author plans to address this in coming work.

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